



## Risk Minimization of Financial Assets Portfolio

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### ABSTRACT

The minimization of the portfolio of financial assets has a particular interest in the field of finance. In this context, several approaches have been proposed to contribute to the solution of this problem which Markowitz approach is the most popular. In this paper, we propose a new approach to minimize the risk of portfolio that measured by a value at risk (VaR) using neural networks. The assets of this portfolio are invested in a market which the fluctuations follow a normal distribution. The minimization procedure is done after the calculation of mathematical explicit formula of VaR using the Black-Scholes stochastic process for these portfolios, which its structure remains constant over the considered time horizon.

**Keywords:** Value at Risk, Neural Networks, Portfolio Risk, Black-Scholes, Stochastic Process, Normal Distribution

**JEL Classifications:** C61, C63, C15

## 1. INTRODUCTION

The minimization of portfolio risk of financial assets is a subject that occupies a particular interest in the field of market risk.

Markowitz was the first that introduce a variance as a measure of portfolio risk of financial assets. But several criticisms were added to this measure because it requires the character of the quadratic objective function and the calculation of the variance-covariance witch making this approach little used in practice.

To remedy this problem several models have been proposed using new risk measures.

In this context, a new risk measure called value at risk (VaR) has been implemented to quantify the maximum loss that might occur with a certain probability, over a given period.

This risk measure is easy to interpret and to compare.

In this article we develop an explicit formula for calculating the VaR for a shares portfolio, then we use this formula to minimize the

VaR of this portfolio using the neural network (NN) (Elhachloufi et al., 2012; Elhachloufi et al., 2012).

This work is organized as follows: Section 1 deals with the portfolio risk. In Section 2, we present the VaR of shares portfolio of normal distribution using Black-Scholes stochastic process. NN are presented in Section 3. Finally, we propose the portfolio minimization procedure.

## 2. RISK PORTFOLIO

The risk of a financial asset is the uncertainty about the value of this asset in an upcoming date. Variance, the average absolute deviation, the semi-variance, VaR and conditional VaR are means of measuring this risk. The portfolio risk is measured by one of the measuring elements mentioned above. It depends on three factors namely:

- The risk of each action included in the portfolio
- The degree of independence of changes in equity together
- The number of shares in the portfolio.

The VaR is defined as the maximum potential loss in value of a portfolio of financial instruments with a given probability over a certain horizon. In simple words, it is a number that indicates how much a financial institution can lose with some probability over a given time. It depends on three elements (Rudd and Rosenbeg 1979; Szergö 2002):

- Distribution of profits and losses of the portfolio that are valid for the period of detention
- Level of confidence
- The holding period of assets.

Analytically, the VaR in time horizon  $t$  and the probability threshold  $\alpha$  is a number  $VaR(t, \alpha)$  such that:

$$P [X \leq VaR(t, \alpha)] = \alpha \tag{1}$$

With:

- $X$  represents the loss (“loss”), is a random variable which might be positive or negative
- $t$  is associated with the VaR horizon which is 1 day for risk metrics or more than a day
- $\alpha$  the probability level is typically 95%, 98% or 99%.

If the distribution of the value of this portfolio is a multivariate normal, then:

$$VaR_\alpha(x) = -x'\mu + z_\alpha \cdot \sqrt{x'\Omega x} \tag{2}$$

Where,

- $\Delta V(x)$  is the value variation
- $\mu = E(\Delta V(x))$  is mean of values
- $\Omega = \sigma(\Delta V(x))$  is standard deviation
- $z_\alpha$  is the quantile of order of confidence  $\alpha$ .

### 3. THE VAR OF SHARES PORTFOLIO OF NORMAL DISTRIBUTION USING BLACK-SCHOLES STOCHASTIC PROCESS

We consider that the price of a share  $S_i$  at time  $t$  is modeled by a stochastic process of Black-Scholes defined by the following stochastic differential equation:

$$dS_i = \mu_i S_i dt + \sigma_i S_i dz \tag{3}$$

Where,

- $\mu$  is the constant drift that indicates the expected return of the share price per unit time;
- $\sigma$  is a constant indicating the annual volatility of the share price.

In discrete case we have

$$\Delta S_i = \mu_i S_i \Delta t + \sigma_i S_i \sqrt{\Delta t} \varepsilon \Rightarrow r_i = \frac{\Delta S_i}{S_i} = \mu_i \Delta t + \sigma_i \sqrt{\Delta t} \varepsilon$$

So for all  $i = 1, \dots, n$  we have:

$$r_i(t) = \mu_i \Delta t + \sigma_i \sqrt{\Delta t} \varepsilon_i$$

$$\text{As } \varepsilon_i \sim N(0,1) \Rightarrow r_i(t) \sim N(\mu_i \Delta t, \sigma_i \sqrt{\Delta t}).$$

The VaR of a portfolio for a horizon  $t$  is noted VaR such as the loss on this portfolio during the  $[0, t]$  not fall below VaR with a fixed probability  $\alpha$ , i.e.,:

$$P[-\Delta V(t) \leq VaR] = \alpha \tag{4}$$

Where,

$$\Delta V(t) = V(t) - V(0) \tag{5}$$

$V(0)$  and  $V(t)$  are respectively the values of portfolio at the beginning and end of the period. More rigorously, the VaR can be defined as:

$$VaR_\alpha = \max \{B/P[-\Delta V(t) \leq B] \leq \alpha\} \tag{6}$$

When the random variable  $\Delta V(T) = V(T) - V(0)$  is distributed according to a normal distribution  $N(E[\Delta V(t)], \sigma[\Delta V(t)])$ , the VaR of probability level  $\alpha$  is defined as follows:

$$P \left[ \frac{\Delta V(t) - E(-\Delta V(t))}{\sigma(\Delta V(t))} \leq \frac{VaR_\alpha - E(-\Delta V(t))}{\sigma(\Delta V(t))} \right] = \alpha$$

If  $\tau_\alpha = \frac{VaR_\alpha - E(-\Delta V(t))}{\sigma(\Delta V(t))}$  is the quantile of the distribution  $N(0,1)$ , we obtain;

$$VaR_\alpha = -E(\Delta V(t)) + \tau_\alpha \sigma(\Delta V(t)) \tag{7}$$

Let  $V(t)$  the value of the portfolio of  $n$  shares invested in a given market at time  $t$ .

We denote by  $x_i$  the number of shares in the portfolio. Let  $S_i(t)$  the price of stock  $i$  at time  $t$ . It follows that:

$$V(t) = \sum_{i=1}^n x_i S_i(t) \tag{8}$$

The portfolio value to the horizon  $T$  is characterized by the following equations:

$$V(T) = \sum_{i=1}^n x_i S_i(T) = \sum_{i=1}^n x_i [S_i(0) + \Delta S_i(T)] \tag{9}$$

By the definition of return  $r_i$  of  $i$  ( $i = 1, \dots, n$ ),

$$r_i(T) = \frac{S_i(T) - S_i(0)}{S_i(0)} = \frac{\Delta S_i(T)}{S_i(0)} \tag{10}$$

The relation (9) becomes:

$$V(T) = \sum_{i=1}^n x_i [S_i(0) + r_i(T) S_i(0)] = \sum_{i=1}^n x_i S_i(0) [1 + r_i(T)].$$

The disadvantage of the equation (6) is that both parameters require knowledge of the univariate parameters  $E(\Delta S_i)$  and  $var(\Delta S_i)$  for each title  $i$  ( $i = 1, \dots, n$ ) and the bivariate parameters  $cov(\Delta S_p, \Delta S_s)$  for each pair of tracks, either in total  $\frac{n(n+1)}{2}$  parameters.

Hence the suggestion of the use of Black-Scholes stochastic process which the simplest and most widely used.

We have:

$$r_i(t) = \mu_i \cdot \Delta t + \sigma_i \cdot \sqrt{\Delta t} \varepsilon_i \tag{11}$$

For all  $i = 1, \dots, n$ ;

So,

$$V(T) = \sum_{i=1}^n x_i S_i(0) [1 + r_i(T)] = \sum_{i=1}^n x_i S_i(0) + \sum_{i=1}^n x_i r_i(T) = V(0) + \sum_{i=1}^n x_i r_i(T)$$

It comes,

$$V(T) - V(0) = \sum_{i=1}^n x_i r_i(T) \Rightarrow \Delta V(T) = \sum_{i=1}^n x_i r_i(T)$$

$$E[\Delta V(T)] = \sum_{i=1}^n x_i E[r_i(T)] = \sum_{i=1}^n x_i \mu_i T$$

and

$$\text{var}[\Delta V(T)] = \text{var}\left[\sum_{i=1}^n x_i r_i(t)\right] = \text{var}\left[\sum_{i=1}^n x_i (\mu_i T + \sigma_i \sqrt{T} \varepsilon_i)\right] = T \left(\sum_{i=1}^n x_i^2 \sigma_i^2\right)$$

Or

$$VaR_\alpha = -E(\Delta V(t)) + \tau_\alpha \sigma(\Delta V(t))$$

Then,

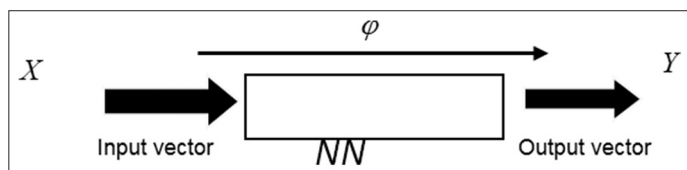
$$VaR_\alpha = -\sum_{i=1}^n x_i \mu_i T + \tau_\alpha \sqrt{T} \left(\sum_{i=1}^n x_i^2 \sigma_i^2\right)$$

### 4. NN

The NN are mathematical models inspired by the structure and behavior of biological neurons. They are composed of interconnected units called artificial neurons capable of performing specific and precise functions. Figure 1 illustrates this situation (Lepage and Solaiman 2003).

For a NN, each neuron is interconnected with other neurons to form layers in order to solve a specific problem concerning the input data on the network.

Figure 1: Black box of neural networks



The input layer is responsible for entering data for the network. The role of neurons in this layer is to transmit the data to be processed on the network. The output layer can present the results calculated by the network on the input vector supplied to the network. Between network input and output, intermediate layers may occur; they are called hidden layers. The role of these layers is to transform input data to extract its features which will subsequently be more easily classified by the output layer.

#### 4.1. Back-propagation Algorithm

The objective of this algorithm is to approximate a function  $y = f(X)$  where  $X$  is an input vector of returns (risk respectively) presented to the input layer assigning each component of  $X$  to a neuron. These inputs are then propagated through the network until they reach the output layer. For each neuron, an activation  $a_i$  is calculated using the formula:

$$a_i = F\left(\sum_j o_j w_{ij}\right) \tag{12}$$

Where,

- $o_j$  is the output of neuron  $j$  of the preceding layer,
- $w_{ij}$  is the weight connecting neuron  $j$  to neuron  $i$ ,
- $F$  is the transfer function (or activation function) of the neuron  $i$ .

The output vector that the network is compared with the product of expected output.

An error  $E$  is calculated as follows:

$$E = \sum_i (o_i - t_i)^2 \tag{13}$$

- $o_i$  is the value neuron output of  $i$  in the output layer
- $t_i$  is the  $i^{\text{th}}$  output target value.

If the error value is not close to 0, the connection weights should be changed to reduce this error. Each weight is either increased or reduced by propagating the error back-calculated. The mathematical formula used by this algorithm is known as the Delta rule:

$$\Delta w_{ij} = \eta \delta_i o_j \tag{14}$$

Where,

- $\Delta w_{ij}$  is the variation weight  $w_{ij}$
- $\eta$  is the learning rate (set by user)
- $\delta_i$  is the error on the output of the neuron  $i$  of a layer.

The calculation depends on the type of neuron. If the neuron is a neuron output, then the error is:

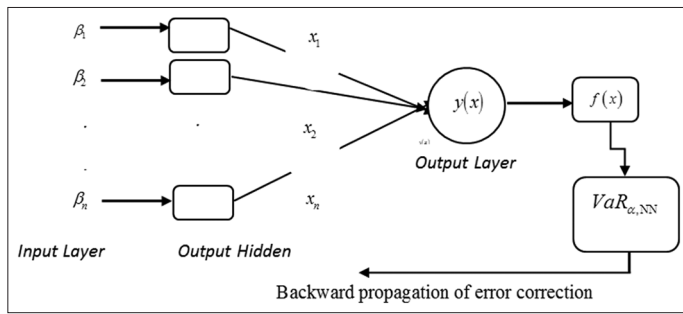
$$\delta_i = F'(a_i)(t_i - o_i) \tag{15}$$

else (hidden neuron)

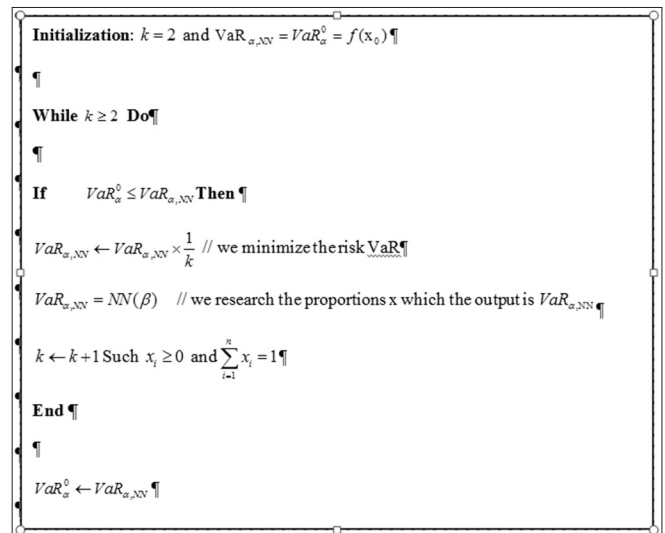
$$\delta_i = F'(a_i) s_k \delta_k w_k \tag{16}$$

Where,  $k$  neurons belonging to the next layer of the neuron  $i$ .

**Figure 2:** Neural network architecture



**Figure 3:** Portfolio minimization risk



The algorithm is repeated for each pair of input/output and more passes are performed until the error has dropped below an acceptable threshold or a maximum number of passes is reached.

In our case, the NN architecture used is an architecture containing a single input layer, one hidden layer composed of  $n$  neurons where  $n$  is the number of  $x_i$  where  $i = 1, \dots, n$  and a layer of containing a single output neuron representing the value of  $VaR_{\alpha, NN}$ .

The learning algorithm used is the gradient back-propagation supervised. The error between the current output (obtained by NNs) and the desired output (observed) spreads, while adjusting the weights with the aim to correct the weights of the network to reduce the global error expressed by the following formula:

$$E = \sum_{i=1}^n (f_i - VaR_{\alpha, NN})^2 \tag{17}$$

Where:

- $f_i$  represents the estimated value of  $f$  in  $i^{th}$  iteration,
- $E$  is the overall error.

The operation of the network illustrated as follows by the Figure 2: Each neuron  $i$  ( $i = 1, \dots, n$ ) in the input layer receives a value of the  $\beta_i$  to be weighted by the proportions of  $x_i$  and the result transmitted to the output layer. In this case, the output  $f$  is given by the following formula:

$$f(x) = -\sum_{i=1}^n x_i \mu_i T + \tau_{\alpha} \sqrt{T} \left( \sum_{i=1}^n x_i^2 \sigma_i^2 \right) = \beta x + y(x) \tag{18}$$

Where,

- $x = (x_1, x_2, \dots, x_n)$
- $\beta = T\mu = (\mu_1, \mu_2, \dots, \mu_n)$  and  $\mu'$  is transposed vector of  $\mu$ .
- $y(x) = \tau_{\alpha} \sqrt{T} X^2 \sigma^2$  where  $X = (x_1^2, x_2^2, \dots, x_n^2)$ ,  $\sigma^2 = (\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2)^t$  and  $T$  is transposed vector of  $I$ .

### 4.2. The Procedure of Portfolio Minimization Risk

The minimization procedure is based on NNs as shown in the Figure 3 (Janssen 2009).

## 5. CONCLUSION

In this paper we presented a new approach to minimize the VaR of a stock portfolio using NNs. This stock portfolio is investing in a market whose fluctuations follow a normal stochastic process.

The price of the stock of portfolio follow the Black-Scholes stochastic process that developed in discrete time assuming that the portfolio structure remains constant over the time horizon.

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